MINIMAL SURFACES IN SIMPLICIAL COMPLEXES

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Prologue

In this paper we introduce the concept of a minimal surface in a simplicial complex.

In the classical case of a smooth minimal surface embedded in a Riemannian manifold the curvature of the surface is bounded from above by the curvature of the manifold. In particular any minimal surface in a manifold of non-positive curvature (for example in a Euclidean manifold) is non-positively curved itself.

We shall define the minimality property in the context of simplicial complexes in such a way, that the minimal surface in a non-positively curved metrical two-dimensional piecewise Euclidean simplicial complex is of non-positive curvature itself. We study the properties of minimal surfaces in simplicial complexes, in particular the existence of such surfaces. Furthermore, we obtain estimates on the Euclidean characteristic of minimal surfaces in some special cases.

This paper is organized as follows. First we consider an arbitrary simplicial complex K with a finite family of disjoint simple closed edge pathes in its 1-skeleton. The "minimal surface problem" in this situation is to find a (minimal in some sence) simplicial complex S whose geometrical realisation is a compact 2-dimensional topological manifold and to find a simplicial map $f : S \to K$, which maps the boundary ∂S exactly onto the given family of edge loops in the 1-skeleton of K.

A simplicial map $f : S \to K$ is the same as a mapping of the vertex set V_S of S to the vertex set K_0 of K fulfilling the following additional property: for any 2-simplex $\{a, b, c\} \subset V_S$ there exists a 2-simplex $\sigma \in K_2$ with $\{f(a), f(b), f(c)\} \subset z$. Let us widen the classes of complexes S and K. Instead of the simplicial complex K we consider an arbitrary set F called a *colour set,* which plays the role of the edge set K_0, together with a subset Z of the set of all triplets of elements of F called the set of *admissible triplets,* which plays the role of K_2.

On the other side let us consider a compact (not necessarily connected) 2-manifold S together with an embedded graph Γ, which plays the role of 1-skeleton in the simplicial settings, fulfilling some additional properties (see Definition 17), which are obviously satisfied in the simplicial case.

Instead of the simplicial map $S \to K$ consider the map $V_\Gamma \to F$ (see Definition 20), which is called *colouring* of the surface S triangulated by the graph Γ.

From now on we consider the triplets (S, Γ, f) consisting of a compact topological surface S, an embedded graph Γ and a proper F-colouring of the vertex set of Γ. Such a triplet is called a (F, Z)-coloured surface. See Definitions 20 and 22 for details.

Definitions 8, 9, 12, and 16 formalize the notion of the boundary condition in the class of coloured triangulated surfaces in both oriented and non-oriented cases. In other words the boundary condition is a finite family (one element for each component of the surface boundary) of finite cyclic sequences of neighbour (in sence of Z) colours from F, where all the colours of given boundary condition are unique.

Now we define a minimal surface as a coloured triangulated surface with a given boundary condition that firstly has the maximal Eulerian characteristic in the class of surfaces with the same boundary condition and secondly the minimal number of edges of the embedded triangulating graph in the class of surfaces with given boundary condition and the maximal Eulerian characteristic. See Definitions 25 and 26 for details.

In the second section we study some properties of the minimal surfaces — see Theorem 27. We show that a minimal surface can not contain certain configurations such as leaves (i.e. vertices of valency one), looped edges (i.e. edges attached to the same vertex with both ends) or multiple edges (i.e. several edges with the same pair of endpoints). These properties show, that a minimal surface is "regular": it is a geometric realisation of some simplicial complex, while the triangulating graph is the 1-skeleton of this complex.

Moreover, two vertices of a minimal surface can not have the same colour if they are contained in the same edge. This property says, that the simplicial map $S \to K$ is "non-degenerated", i.e. the image of any simplex has the same dimension as the pre-image. This behaviour is analogous to the embedding property for smooth maps, it allows to induce a metric on the pre-image space using the metric on the target space as done in Section 5.

The last property of Theorem 27 says that the opposite vertices of a pair of triangles with a common edge can not have the same colour. It follows that the locally Euclidean metric induced on the minimal surface in a 2-dimensional complex of non-positive curvature is of non-positive curvature itself, see Section 5 for details.

The proof of these properties is by explicite construction. If a coloured triangulated surface does not satisfy one of these properties, we construct from this surface by means of contracting subsets, cutting subsets out or gluing another coloured triangulated surface which is less complex in the sence of Definition 25. Then the construction implies that the original coloured triangulated surface is not minimal.

In the third section we study the existence problem for minimal surfaces in simplicial complexes. The existence question in the smooth case is a rather complicated problem. In out combinatorial setting it is almost trivial. Because of the discretness of the complexity function (see Definition 25) there exists a minimal surface with given boundary condition if and only if there exists at least one triangulated coloured surface with the same given boundary condition. Theorem 46 states that the existence of a minimal surface in a simplicial complex with given boundary condition is equivalent to the purely topological property of the family of boundary loops to be a zero co-bordant family.

In the forth section we construct a map from the space of finite loop families in a topological space into the first homology group of the space (with coefficients in $\mathbb{Z}/2\mathbb{Z}$ in the non-oriented case). Then we prove that the loop family considered in the third section is zero co-bordant if and only if its image in an appropriate homology group vanishes. We prove these statements in both oriented and non-oriented cases.

In the fifth section we discuss the applications of out results to two-dimensional simplicial complexes with a piecewise Euclidean metrics of non-positive curvature. We first prove an inequality among the sum of link lengths over all boundary vertices, the number of boundary edges and the Eulerian characteristic. We then use this to estimate the Eulerian characteristic in several special cases (Theorem 71).

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1. Basic definitions

Definition 1. Let $\{A_i \mid i \in I\}$ be an indexed family of sets. By $\sqcup_{i\in I} A_i$ we denote the set $\{(i, a) \mid i \in I, a \in A_i\}$.

Definition 2. Let a and b be two integer numbers. By $[a, b]$ we denote the set $\{x \in \mathbb{Z} \mid a \leqslant x \leqslant b\}$, which can be empty. In some rare cases we shall denote by the same symbol the interval

$$\{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}$$

in the set of real numbers.

Definition 3. Let X be a finite set. By $\#X$ we denote the number of elements of X.

Definition 4. Let K be a simplicial complex, $i \geqslant 0$ an integer. By K_i denote the set of simplicies of K of dimension i. By $|K|$ denote the geometric realisation of the complex K. For a simplex $\sigma \in K$ denote by $|\sigma|$ the subset of $|K|$ defined as the convex hull of the vertices of σ. By $|K|_i$ denote the i-th skeleton of K as a subset of $|K|$:

$$|K|_i = \bigcup_{\substack{\sigma\in K \\ \dim \sigma = i}} |\sigma|.$$

Definition 5. Let N be a non-negative integer. Let K be a non-empty simplicial complex. Let us call K *thick* N-*dimensional* (or just N-*thick*) complex if every simplex in K is contained in an N-dimensional simplex in K.

Remark 6. The dimension of an N-thick complex is automatically equal to N.

Definition 7. Let X and Y be two topological spaces. By $C(X, Y)$ we denote the set of all continuous maps from X to Y.

Definition 8. Let F be an arbitrary set. Let $\operatorname{Seq} F$ be the set of all triplets (k, a, f), where

(1) k is a natural number;
(2) a is a map $[1, k] \to \{x \in \mathbb{N} \mid x \geqslant 3\}$;
(3) f is a map $\sqcup_{i=1}^{k} [0, a(i) - 1] \to F$.

$\operatorname{Seq} F$ is the set of finite families of finite sequences (of length at least 3) in F. Let us denote an element $(k, a, f) \in \operatorname{Seq} F$ by the k-tuple of sequences of elements of F: $(k, a, f) = \big(f(1, 0) \cdots f(1, a(1) - 1), \dots, f(k, 0) \cdots f(k, a(k) - 1)\big)$.

Definition 9. Let F be a set, $s = (k, a, f)$ and $s' = (k', a', f')$ two elements of $\operatorname{Seq} F$. We shall say that s and s' are oriented equivalent (and denote this by $s \overset{\circ}{\sim} s'$) if the following conditions hold:

(1) $k = k'$;
(2) there exists a permutation $\tau \in S_k$ such that $a' = a \circ \tau$;
(3) there exist integer numbers b_i $(i \in [1, k])$ such that

$$f'(i, n) = f(\tau(i), n + b_i \mod a'(i))$$

for any $n \in [0, a'(i) - 1]$.

We shall say that s and s' are equivalent (and write $s \sim s'$) if together with properties (1) and (2) the following weaker condition holds:

(3′) for any $i \in [1, k]$ there exist integers $c_i \in \mathbb{Z}$ and $\epsilon_i \in \{-1, +1\}$ such that

$$f'(i, n) = f(\tau(i), \epsilon_i n + c_i \mod a'(i))$$

for any $n \in [0, a'(i) - 1]$.

Hence two families of sequences from $\operatorname{Seq} F$ are oriented equivalent if and only if they consist of the same sequences up to permutation of sequences and shifts in the sequences. They are equivalent if we also allow for the shifts to be preceeded by reflections in some sequences.

Proposition 10. *The both relations* $\overset{\circ}{\sim}$ *and* $\sim$ *are equivalence relations. Suppose that* $s, s' \in \operatorname{Seq} F$ *satisfy* $s \overset{\circ}{\sim} s'$*, then* $s \sim s'$*.*

Proof. The first claim is obvious. To proof the second claim set $c_i = b_i$ and $\epsilon_i = +1$. □

Example 11. *Let* F *be the finite set* $\{a, b, c, d, e, f, x, y, z\}$*. By* s *denote the element* $(dyf, ezx) \in \operatorname{Seq} F$*. By* s' *denote* $(xze, yfd) \in \operatorname{Seq} F$*. It holds* $s \sim s'$*, but not* $s \overset{\circ}{\sim} s'$*.*

Definition 12. For any set F denote the quotient sets $\operatorname{Seq} F/\sim$ (resp. $\operatorname{Seq} F/\overset{\circ}{\sim}$) by $B(F)$ (resp. by $B^o(F)$).

Definition 13. Let F be a set. Let F_3 be the set of all 3-element subsets of F. Let Z be some subset of F_3. Let us call the pair (F, Z) a *colouring system*, the elements of F *colours*, and the elements of Z *admissible triplets.*

Definition 14. Let (F, Z) be a colouring system. Let us call an element $(k, a, f) \in \operatorname{Seq} F$ *admissible,* if the following conditions hold:

(1) for any $i \in [1, k]$ and $n \in [0, a(i) - 1]$ there exists $z \in Z$ such that
$$\{f(i, n), f(i, n+1 \mod a(i)\} \subset z,$$
i.e. all the pairs of colours appearing as consequent elements in one of the sequences of (k, a, f) are contained in admissible triplets;

(2) the mapping f is injective.

By $\mathrm{Seq}(F, Z)$ denote the subset of all admissible elements of $\mathrm{Seq}\, F$.

Proposition 15. *Let* (F, Z) *be a colouring system. Let* s, s' *be elements of* $\mathrm{Seq}\, F$. *If* s *is admissible and* $s \overset{\circ}{\sim} s'$ *(or even* $s \sim s'$*), then* s' *is also admissible.*

Proof. The proof is trivial. □

Definition 16. Let (F, Z) be a colouring system. Denote by $B(F, Z)$ (resp. $B^o(F, Z)$) the subset of $B(F)$ (resp. $B^o(F)$) consisting of equivalence classes of admissible elements of $\mathrm{Seq}\, F$. Because of Proposition 15 the sets $B^o(F, Z)$ and $B(F, Z)$ are well-defined. Because of Proposition 10 there exists a natural projection $\beta_{F,Z} : B^o(F, Z) \to B(F, Z)$.

Definition 17. Let S be a compact (not necessary connected) two-dimensional manifold. For any component S_1 of S let the boundary ∂S_1 be a non-empty set. Let $\Gamma \subset S$ be an embedded graph. Let us call the pair (S, Γ) *triangulated surface,* if the following conditions hold:

(1) the boundary ∂S of the surface is contained in Γ;
(2) each component of $S \backslash \Gamma$ is a topological 2-disc;
(3) each component of $S \backslash \Gamma$ is bounded by a cycle of edges of Γ of length at most 3;
(4) each component of ∂S is a cycle of edges of Γ of length at least 3.

Definition 18. Let (S, Γ) be a triangulated surface. Let V be the vertex set of Γ. Let F be an arbitrary set. Let us call a map $f : V \to F$ an F-*colouring* of the triangulated surface (S, Γ).

Definition 19. Let (S, Γ) and $f : V \to F$ be as above. Choose first a numbering of boundary components of S by integers from $[1, k]$, where $k = \#\pi_0(\partial S)$. Then choose for any component C_i of ∂S a numbering of the vertices from $C_i \cap V$ by integers from $[0, a(i) - 1]$ (where $a(i)$

is defined as $\#(C_i \cap V)$) in such a way that the consequent vertices are numbered by consequent (modulo $a(i)$) integers. By $\nu(i, n)$ denote the n-th vertex on the i-th component of ∂S. By s denote the element $(k, a, f \circ \nu)$ of $\operatorname{Seq} F$. The element s depends on the choice of the component and vertex numberings, but its class $[s] \in B(F)$ does not. Let us call $[s] \in B(F)$ the *boundary condition* for the coloured triangulated surface (S, Γ, f) and denote it by $\partial(S, \Gamma, f)$. Let S be now an oriented surface, and we suppose that the vertex numbering respects the orientation of S. In this case the class $[s] \in B^o(F)$ does not depend on the both choices. Let us call this class the *oriented boundary condition* for (S, Γ, f) and denote it by $\partial^o(S, \Gamma, f)$.

Definition 20. Let (S, Γ) be a triangulated surface, V the vertex set of Γ, (F, Z) a colouring system. Let us call a map $f : V \to F$ an (F, Z)-*admissible colouring,* if the following conditions hold:

(1) for any component U of $S \backslash \Gamma$ there exists an admissible triplet $z \in Z$, which contains the boundary vertices of U: $f(V \cap \bar{U}) \subset z$;
(2) for any two vertices on the surface boundary $x, y \in \partial S \cap V$ it holds $f(x) \neq f(y)$, i.e. the colouring map is injective on the surface boundary.

Proposition 21. *Let* (S, Γ) *be a triangulated surface,* (F, Z) *a colouring system. If* f *is an* (F, Z)*-admissible colouring of* (S, Γ)*, then* $\partial(S, \Gamma, f) \in B(F, Z)$*. In the case of oriented surface* S *we have* $\partial^o(S, \Gamma, f) \in B^o(F, Z)$.

Definition 22. Let (S, Γ) be a triangulated surface, (F, Z) a colouring system. If f is a (F, Z)-admissible colouring of (S, Γ), then the triplet (S, Γ, f) is called an (F, Z)-*coloured surface.*

Definition 23. Let us define the *lexicographical ordering* on the set $\mathbb{Z}^2$ of pairs of integers by

$$(x_1, y_1) \leqslant (x_2, y_2) \iff x_1 < x_2 \vee (x_1 = x_2 \wedge y_1 \leqslant y_2).$$

Definition 24. By $\chi_1 \Gamma$ denote the number of edges of the graph Γ.

Definition 25. Let (S, Γ) be a triangulated surface (either oriented or not), χS the Eulerian characteristic of the surface S. Call the pair $\kappa(S, \Gamma) = (-\chi S, \chi_1 \Gamma)$ (considered as an element of the lexicoraphically ordered set $\mathbb{Z}^2$) the *complexity* of the triangulated surface (S, Γ).

Definition 26. Let (F, Z) be a colouring system. Let (S, Γ, f) be an (F, Z)-coloured triangulated surface (either oriented or not). It is called *minimal* if the inequality $\kappa(S, \Gamma) \leqslant \kappa(S', \Gamma')$ holds for any (F, Z)-coloured surface (S', Γ', f') with

- $\partial^o(S, \Gamma, f) = \partial^o(S', \Gamma', f')$ in oriented case, or
- $\partial(S, \Gamma, f) = \partial(S', \Gamma', f')$ in non-oriented case.

2. Regularity of minimal surfaces

Theorem 27. *Let* (F, Z) *be a colouring system. Let* (S, Γ, f) *be a minimal* (F, Z)*-coloured triangulated surface (either oriented or not). Then the following properties hold:*

1. *Every vertex of* Γ *is an end-point of at least two edges.*
2. *The graph* Γ *does not have loops, i.e. edges whose ends coincide.*
3. *If two vertices* a_1 *and* a_2 *of* Γ *are connected by an edge, then* $f(a_1) \neq f(a_2)$.
4. *For any two vertices of* Γ *there is at most one edge connecting them.*
5. *Let* b *and* c *be two vertices of* Γ *connected by an edge. Let* a_1 *and* a_2 *be two other vertices, and assume that both of them are connected to* b *as well as to* c. *Assume that there are components* U_1 *resp.* U_2 *of* $S \backslash \Gamma$ *spanning the triangles* (a_1bc) *resp.* (a_2bc). *Then* $f(a_1) \neq f(a_2)$.

Definition 28. Let (S, Γ) be a triangulated surface. It is called *simplicial,* if there exists a simplicial complex X and a homeomorphism $h : S \to |X|$ such that $h(\Gamma)$ is the 1-skeleton of X.

Corollary 29. *Any minimal triangulated surface is simplicial.*

Proof. The claim follows from the statements (1)–(4) of Theorem 27. □

In the remaining part of this section we shall prove Theorem 27. The proof will be done by reductio ad absurdum. For any of the five properties we shall assume the converse and construct some other (F, Z)-coloured surface (S_0, Γ_0, f_0) with the same boundary condition as for (S, Γ, f) and with smaller complexity: $\kappa(S_0, \Gamma_0) < \kappa(S, \Gamma)$. The construction of S_0 will be done by cuttings along some edges of the graph Γ and consequent gluing along some edge pairs with matching end-point colouring. The colouring of the new vertices will be inherited from the vertices of Γ.

Definition 30. Let M be a topological manifold. Suppose J be a subset of M homeomorphic to $\mathbb{S}^1$. Let $\pi : M_2 \to M$ be the orientation covering of M. The pre-image $\pi^{-1}J \subset M_2$ of the loop J is either a single loop in M_2 or the union of two disjoint loops. In the first case we shall call the loop J *orientation inversing*. In the second case it is called *orientation preserving* loop.

Remark 31. Let M be a surface, i.e. topological manifold of dimension 2 and J a loop as above. Let U be a tubular neighbourhood of J. The loop J is orientation inversing if and only if U is homeomorphic to a Möbius strip.

2.1. **At least two edges at a vertex.** Let a be a vertex of Γ, let e be the only edge of Γ connected to a. Let S_0 be the same surface as S. Remove both a and e from the graph Γ and call the resulting graph Γ_0. Let f_0 be the restriction of f to the vertex set of Γ_0. We have $\chi_1\Gamma_0 = \chi_1\Gamma - 1$ and thus $\kappa(S_0, \Gamma_0) < \kappa(S, \Gamma)$. This proves the first statement of Theorem 27.

2.2. **No loops.** Let a be a vertex of Γ, let e be an edge connecting a to itself. Let us cut the surface S along the edge e and call the resulting surface S_1. We have four topologically different cases depending on two parameters: either or not the edge e is orientation preserving, and either or not $a \in \partial S$ (in any case $e \not\subset \partial S$ by property (4) of triangulated surfaces). Let a_i (where $i \in [1,2]$ for $a \notin \partial S$, and $i \in [1,3]$ for $a \in \partial S$) be the new vertices of $\Gamma_1 \subset S_1$. Let e_1 and e_2 be the new edges. See Figure 1 for details.

Let us construct the surface S_0. If the edge e is orientation inversing (the cases 2 and 4) just contract the edges e_1 and e_2 and all the vertices a_i to a new vertex a_0. Let us call the resulting surface S_0. In the case 2 we have $\chi S_0 = \chi S_1 = \chi S + 1$. In the case 4 we have $\chi S_0 = \chi S_1 + 1 = \chi S + 1$. This proves the second statement of the theorem for the case of orientation inversing edge e.

Consider the case 3. For $i \in \{1,2\}$ let us contract the edge e_i together with the vertex a_i to a new vertex a_i'. Let us call the resulting surface S_0'. We have $\chi S_0' = \chi S_1 + 2 = \chi S + 2$. It is possible that the points a_1' and a_2' belong to two different components of S_0', and one of these two components (but not the both of them) is a closed surface. In this case define S_0 by removing this closed component of the surface S_0'. Because the Eulerian characteristic of a surface can not be greater than 2, it follows from $\chi S_0' = \chi S + 2$, that $\chi S_0 \geqslant \chi S$.

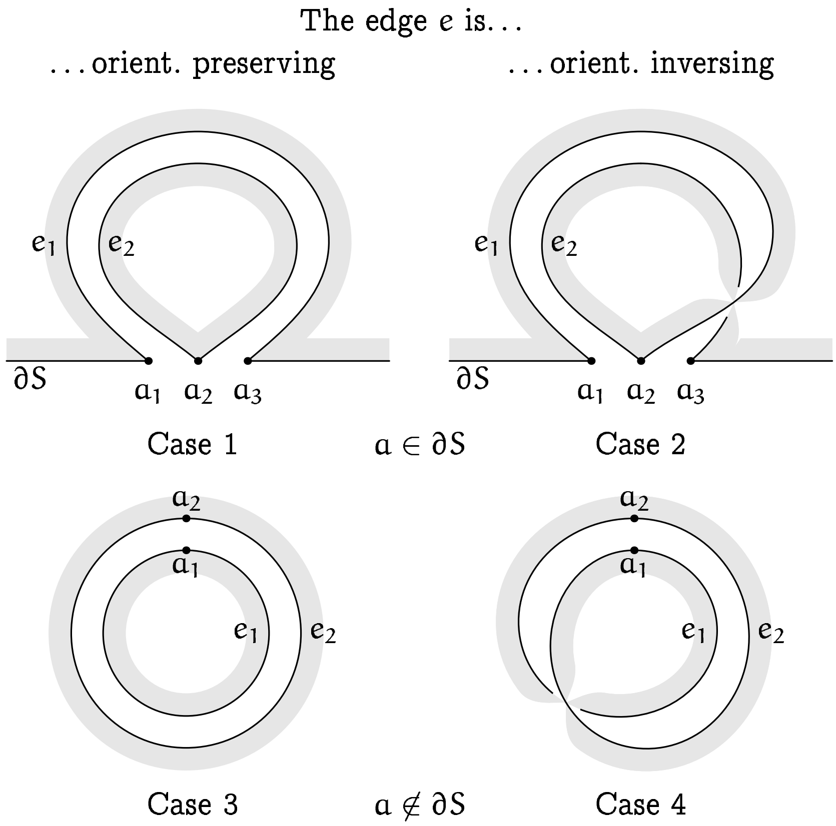


FIGURE 1. The surface S cutted along the loop edge e.

Therefore $\kappa(S_0, \Gamma_0) < \kappa(S, \Gamma)$, as $\chi_1\Gamma_0 < \chi_1\Gamma$. In other case (none of a_i' belongs to a closed component) just define $S_0 = S_0'$. This concludes the proof for the case 3.

As for the case 1, let us contract a_1, a_3, e_1 to a_1', and a_2, e_2 to a_2'. After the optional removing of the closed component containing a_2', it defines (analogous to the case 3) an appropriate surface S_0. This concludes the proof of the second statement of the Theorem.

2.3. **Neighbours have different colours.** Let an edge e of the graph Γ connect two vertices a_1, a_2 with $f(a_1) = f(a_2)$. As the vertices a_1 and a_2 can not both belong to ∂S (because of $f(a_1) = f(a_2)$), we may assume that $a_2 \notin \partial S$. Let us define the graph Γ_0 by removing the vertex a_2 and the edge e and connecting the loose ends of the edges directly to the vertex a_1, as shown in Figure 2. Obviously we have $\chi_1\Gamma_0 = \chi_1\Gamma - 1$. The proof of the third statement of the Theorem is now completed by defining S_0 to be S.

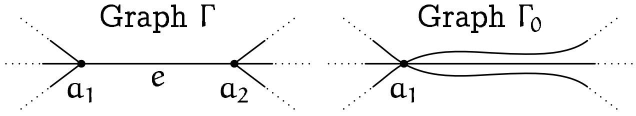


FIGURE 2. Contraction of an edge connecting two vertices of the same colour.

2.4. No double edges. Let a and b to be two different vertices of Γ connected by two distinct edges e_1 and e_2. In order to construct the surface S_0 we consider four different cases:

(i) $a \in \partial S$, $b \in \partial S$, $e_1 \not\subset \partial S$, $e_2 \not\subset \partial S$;
(ii) $a \in \partial S$, $b \in \partial S$, $e_1 \subset \partial S$, $e_2 \not\subset \partial S$;
(iii) $a \in \partial S$, $b \notin \partial S$;
(iv) $a \notin \partial S$, $b \notin \partial S$.

The case (i): $a, b \in \partial S$, $e_1, e_2 \not\subset \partial S$. Consider a small neighbourhood of the point a. The edges e_1 and e_2 cut it in three areas: α_0 adjacent to both e_1 and e_2, and α_j (for $j \in \{1, 2\}$) adjacent to e_j only. Denote by β_0, β_1, β_2 the analogous areas in the neighbourhood of b — see Figure 3 for details.

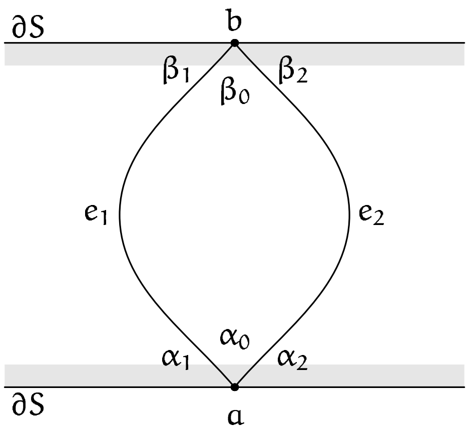
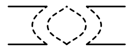


FIGURE 3. Case (i): two inner edges connecting the same pair of boundary vertices.

Let us cut the surface S along the edges e_1 and e_2, and call the resulting surface S_1. Denote by a_j resp. b_j (for $j \in \{0, 1, 2\}$) the new vertices adjacent to α_j resp. β_j. Denote by e_i' and e_i'' two new boundary edges in place of e_i for $i \in \{1, 2\}$. Choose these notations in such a way that $b_0 \in e_1'' \cap e_2''$, $b_1 \in e_1'$, $b_2 \in e_2'$. For every pair (e_j', e_j''), where $j \in \{1, 2\}$, there are two possible connections between $\{a_0, a_j\}$ and $\{b_0, b_j\}$. These two possibilities are denoted by dotted resp. dashed lines in Figure 4. Thus we have to consider four subcases:

(i_1) $a_1 \in e_1'$, $a_0 \in e_1''$; $a_2 \in e_2'$, $a_0 \in e_2''$; 

(i_2) $a_0 \in e_1'$, $a_1 \in e_1''$; $a_0 \in e_2'$, $a_2 \in e_2''$; 

(i_3) $a_1 \in e_1'$, $a_0 \in e_1''$; $a_0 \in e_2'$, $a_2 \in e_2''$; 

(i_4) $a_0 \in e_1'$, $a_1 \in e_1''$; $a_2 \in e_2'$, $a_0 \in e_2''$; 

For symmetry reasons the subcase (i_4) follows from the subcase (i_3), therefore we shall consider only the first three subcases.

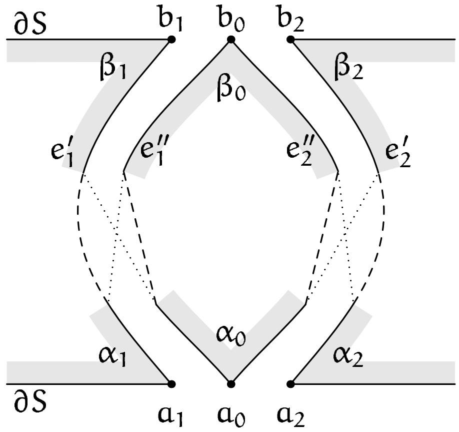


FIGURE 4. The surface S cutted along two inner edges connecting the same pair of boundary vertices.

The subcase (i_1). Define the surface S_0' by identification of the edges e_1' with e_2' and e_1'' with e_2''. Denote by a_{12} resp. b_{12} the common image of a_1 and a_2 resp. b_1 and b_2 (see Figure 5). Now proceed as in the cases 1 and 3 of 2.2: remove the component containing a_0 if it is a closed surface. Call the resulting surface S_0. If we have removed a component, we have $\chi S_0 = \chi S$, and $\chi_1 \Gamma_0 \leqslant \chi_1 \Gamma - 1$. In the other case ($S_0 = S_0'$) we have $\chi S_0 = \chi S + 2$. Thus $\kappa(S_0, \Gamma_0) < \kappa(S, \Gamma)$.

The subcase (i_2). Let us define the surface S_0 by identification of edges e_1' to e_2' and e_1'' to e_2''. Denote by a_{12} resp. b_{12} the common image of a_1 and a_2 resp. b_1 and b_2 (see Figure 6 for details). We have $\kappa(S_0, \Gamma_0) < \kappa(S, \Gamma)$, as $\chi S_0 = \chi S + 2$.

The subcase (i_3). Let us define the surface S_0 by identification of edges e_1'' to e_2' and e_1' to e_2''. Denote by a_{012} the common image of a_0, a_1 and a_2, by b_{12} the common image of b_1 and b_2 (see Figure 7 for details). We have $\kappa(S_0, \Gamma_0) < \kappa(S, \Gamma)$, as $\chi S_0 = \chi S + 1$. This concludes the proof in the case (i).

The cases (ii), (iii), and (iv). Denote by J the loop $e_1 \cup e_2$ and cut the surface S along the loop J (in the case (ii) along the edge e_2 only, since the edge e_1 already belongs to the boundary ∂S).

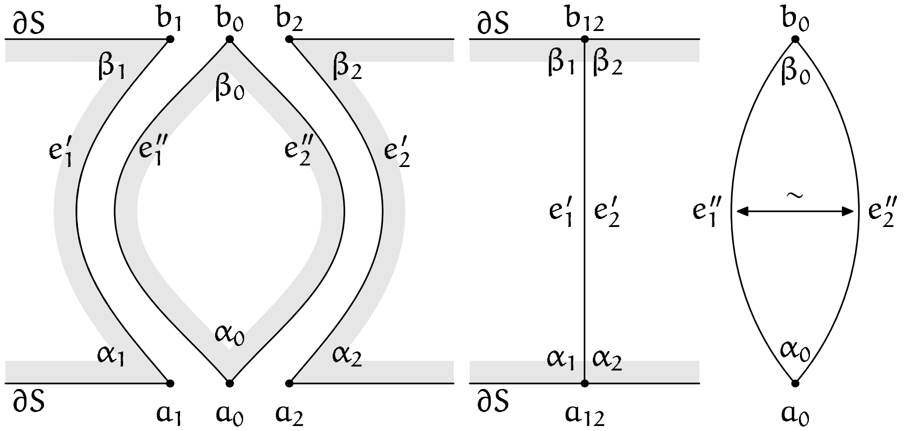


FIGURE 5. Case (i_1): the surface cutted along e_1, e_2 and glued by indentifications $e'_1 \sim e'_2$, $e''_1 \sim e''_2$.

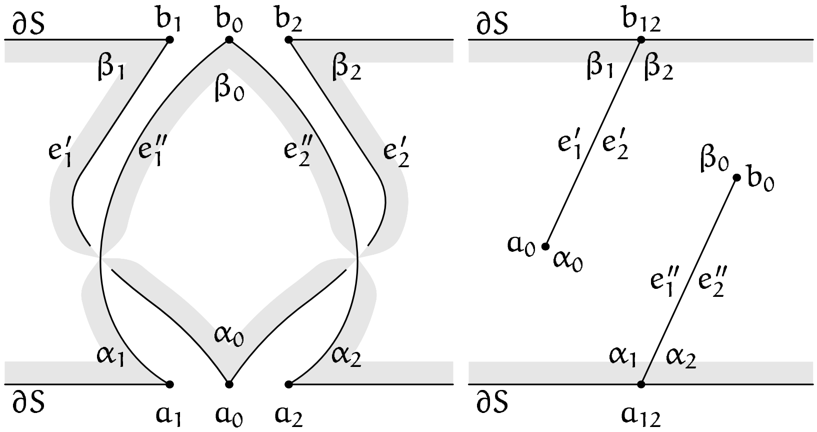


FIGURE 6. Case (i_2): the surface cutted along e_1, e_2 and glued by indentifications $e'_1 \sim e'_2$, $e''_1 \sim e''_2$.

The loop J is either orientation inversing or orientation preserving. Let us first assume that J is orientation preserving. In the cases (iii) and (iv) identify the edges $e'_1 \sim e'_2$ and $e''_1 \sim e''_2$ (see Figure 8 for details). In the case (ii) just identify $e_1 \sim e'_2$ (see Figure 9). After the gluing remove a closed component if it occures. Call the resulting surface S_0. After removing a closed component, it holds $\chi S_0 \geqslant \chi S$ and $\chi_1 \Gamma_0 < \chi_1 \Gamma_1$. If there was no closed component to remove, we have $\chi S_0 = \chi S + 2$ (in the case of orientation preserving J).

Now assume that J is orientation inversing. In the case (ii) identify $e'' \sim e_1$ (see Figure 10). In the cases (iii) — see Figure 11 — and (iv) — see Figure 12 — identify $e'_1 \sim e''_2$ and $e''_1 \sim e'_2$. Call the resulting

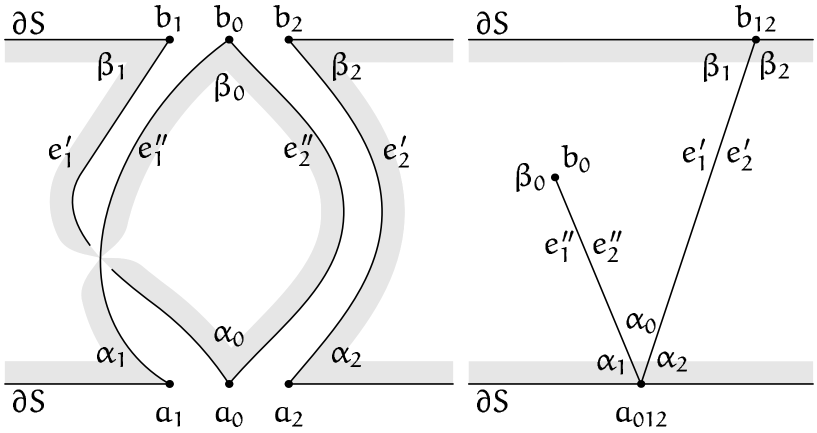


FIGURE 7. Case (i_3): the surface cutted along e_1, e_2 and glued by indentifications $e_1' \sim e_2'$, $e_1'' \sim e_2''$.

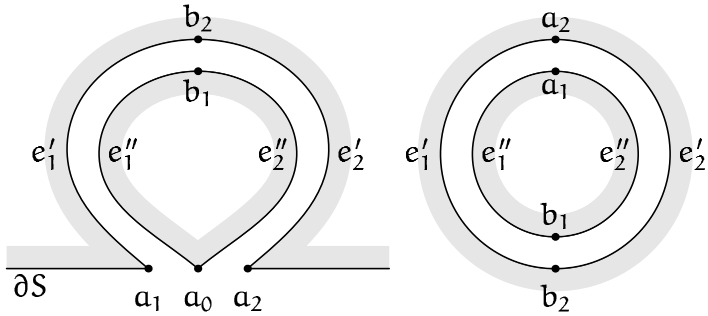


FIGURE 8. Cases (iii) and (iv): the surface cutted along two inner edges with orientation preserving $e_1 \cup e_2$.

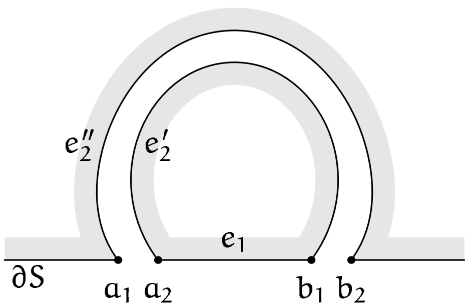


FIGURE 9. Case (ii): the surface cutted along an inner edge with boundary end-points; $e_1 \cup e_2$ is orientation preserving.

surface S_0. It holds $\chi S_0 = \chi S + 1$. This concludes the proof of the 4-th statement of the Theorem.

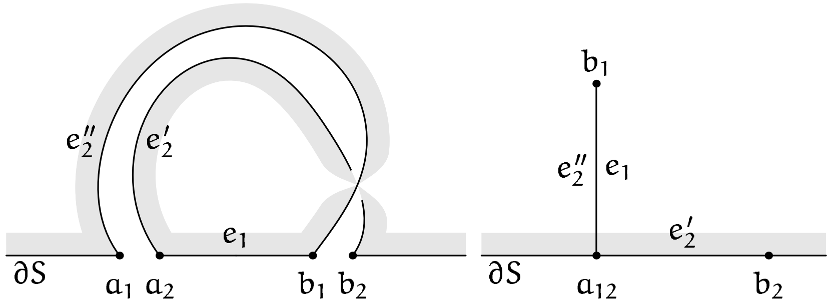


FIGURE 10. Case (ii): gluing in the orientation inversing subcase.

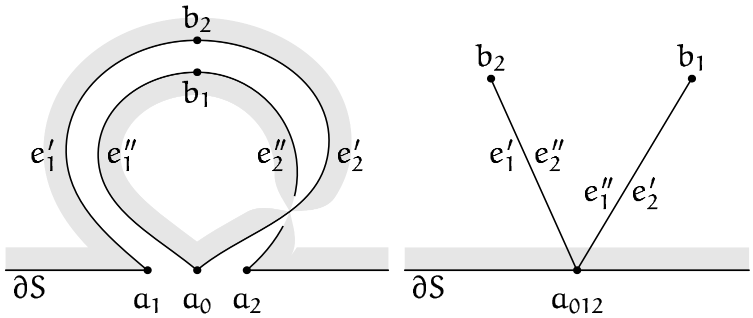


FIGURE 11. Case (iii): gluing in the case of orientation inversing J.

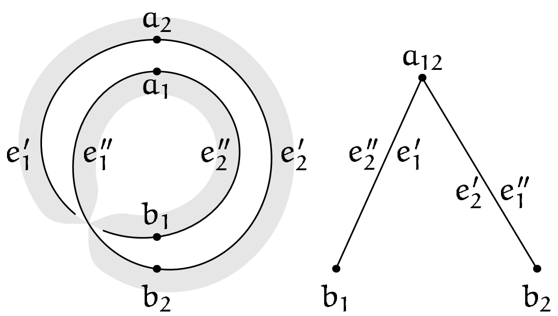


FIGURE 12. Case (iv): gluing in the case of orientation inversing J.

2.5. **The opposite vertices of adjacent triangles differ.** Assume that in the setting of the 5-th statement of the Theorem we have $f(a_1) = f(a_2)$. Remove the interior of the triangles (a_1bc) and (a_2bc), remove the edge (bc) and identify the edges (a_1b) to (a_2b) and (a_1c) to (a_2c) (see Figure 13). The resulting surface S_0 is homeomorphic to S and we have $\chi_1\Gamma_0 = \chi_1\Gamma - 3$. This completes the proof of Theorem 27.

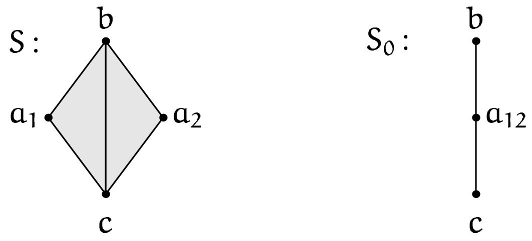


FIGURE 13. Cutting out adjacent triangles.

3. EXISTENCE OF MINIMAL SURFACES

Definition 32. Let X be a topological space. By PX denote the set $\sqcup_{k=1}^{\infty} C(\sqcup_{i=1}^{k} \mathbb{S}^1, X)$ of finite sequences of loops in the space X.

Definition 33. For two elements $p_1 = (k_1, \phi_1), p_2 = (k_2, \phi_2)$ in PX denote by $p_1 \sqcup p_2$ the the element $(k_1 + k_2, \phi_{12}) \in PX$, where the map ϕ_{12} maps the first k_1 circles as ϕ_1 does, and the remaining k_2 circles are mapped according to ϕ_2.

Definition 34. For an element $p = (k, \phi) \in PX$ denote by p^{-1} the element $(k, \phi') \in PX$, where the map ϕ' is defined by $\phi'(i, \zeta) = \phi(i, \zeta^{-1})$ for any $i \in [1, k]$, $\zeta \in \mathbb{S}^1 \subset \mathbb{C}$.

Definition 35. Let X be a topological space, (k, ϕ) and (k', ϕ') two elements of PX. Assume that $k = k'$ and there exists a homeomorphism ψ of the space $\sqcup_{i=1}^{k} \mathbb{S}^1$ to itself such that $\phi' = \phi \circ \psi$. Then the elements (k, ϕ) and (k', ϕ') are called *equivalent.* We denote this by $(k, \phi) \sim (k', \phi')$. If the homeomorphism ψ can be choosen orientation preserving, these elements are also called *oriented equivalent.* We denote this by $(k, \phi) \overset{\circ}{\sim} (k', \phi')$.

Definition 36. Let X be a topological space. Let LX denote the quotient space $PX/\sim$. Let us call the elements of LX *(finite) loop families* in X. Let us denote by $L^{o}X$ the space $PX/\overset{\circ}{\sim}$ of *oriented* finite loop families in X. Since the relation $\overset{\circ}{\sim}$ implies the relation $\sim$, there is a natural projection $\lambda_X : L^{o}X \to LX$. The operators $\sqcup$ and $\cdot^{-1}$ defined on PX clearly can be pushed down to the quotients LX and $L^{o}X$.

Proposition 37. *Let* X *be a topological space. The operator* $\sqcup$ *is commutative on both* LX *and* $L^{o}X$*. For* $z \in LX$ *we have* $z^{-1} = z$*, but for an element of* $L^{o}X$ *this equation holds only in trivial cases.*

Definition 38. Let K be a simplicial complex, consider the colouring system (K_0, K_2): the colours are the vertices of K, the admissible triplets are the vertex sets of all 2-simplicies of K. Let $s = (k, a, f)$ be an element of $\mathrm{Seq}(K_0, K_2)$. Let $\tilde{l}(s)$ be a continuous map from $\sqcup_{i=1}^{k} \mathbb{S}^1$ to $|K|$ such that the restriction of this map to the i-th circle is a homeomorphism on the image, the image is the closed path $f(0), f(1), \dots, f(a(i)-1), f(0)$ in $|K|$ and this path is parametrized by the arc length. This path is well defined: the vertices $f(j)$ and $f(j+1)$ do not coincide and are connected by an edge in K_1 because s is (K_0, K_2)-admissible.

Proposition 39. *Let* K *be a simplicial complex,* $s_1, s_2 \in \mathrm{Seq}(K_0, K_2)$*. If* $s_1 \sim s_2$ *(resp.* $s_1 \overset{\circ}{\sim} s_2$*), then* $\tilde{l}(s_1) \sim \tilde{l}(s_2)$ *(resp.* $\tilde{l}(s_1) \overset{\circ}{\sim} \tilde{l}(s_2)$*) in* $P|K|$*.*

Definition 40. Let K be a simplicial complex. By l_K denote the map $B(K_0, K_2) \to L|K|$ taking an equivalence class $[s]$ to $[\tilde{l}(s)]$. Analogously define the map $l_K^o : B^o(K_0, K_2) \to L^o|K|$.

Proposition 41. *Let* K *be a simplicial complex. Then the maps* l_K *and* l_K^o *are injective and the following diagram commutes:*

$$\begin{array}{ccc} B^o(K_0, K_2) & \xrightarrow{l_K^o} & L^o|K| \\ {\scriptstyle \beta_{K_0,K_2}} \downarrow & & \downarrow {\scriptstyle \lambda_{|K|}} \\ B(K_0, K_2) & \xrightarrow{l_K} & L|K| \end{array}$$

Proposition 42. *If* K *is a thick simplicial complex of dimension at least 2, then the images of the maps* l_K *and* l_K^o *consist exactly of the families of distinct closed simple pathes in the graph* $|K|_1$*.*

Definition 43. Let X be a topological space. An element $(k, \phi) \in PX$ is called *zero co-bordant* if there exists a compact two-dimensional manifold S without closed components, a homeomorphism $\tau : \partial S \to \sqcup_{i=1}^{k} \mathbb{S}^1$, and a continuous map $f : S \to X$ satisfying $f|_{\partial S} = \phi \circ \tau$. The map f is called *co-bordism.* If the surface S is oriented and the map τ is orientation preserving, the element (k, ϕ) is called *oriented zero co-bordant.* Because the property to be zero co-bordant (resp. oriented zero co-bordant) depends only on the equivalence class of the element with respect to the relation $\sim$ (resp. $\overset{\circ}{\sim}$), this property is well defined on the set LX (resp. L^oX).

Example 44. *A family* x *consisting of a simple null-homotopic curve is oriented zero co-bordant. For any family* z *the element* $z \sqcup z^{-1}$ *is oriented zero co-bordant, while the element* $z \sqcup z$ *is zero co-bordant, but not necessarily oriented zero co-bordant.*

Definition 45. Let X be a topological space. Let $x \in LX$ be a zero co-bordant loop family. It is called *minimal* if there are no zero co-bordant elements $y, z \in LX$ with $x = y \sqcup z$. Obviously if $f : S \to X$ is a co-bordism for x then the surface S is connected. We shall say that x is minimal zero co-bordant of characteristic χ if

$$\chi = \max\left\{\chi S \mid f : S \to X \text{ is a co-bordism for } x\right\}$$

Obviously it holds $\chi \leqslant 1$.

Theorem 46. *Let* K *be a simplicial complex. Let* $[s] \in B(K_0, K_2)$ *be an (oriented) boundary condition. Then there exists an (oriented)* (K_0, K_2)*-coloured triangulated surface* (S, Γ, f) *satisfying the given boundary condition* $\partial(S, \Gamma, f) = [s]$ *if and only if the element* $l_K[s] \in L|K|$ *is (oriented) zero co-bordant.*

Proof. We shall prove the non-oriented case of the theorem, the proof for the oriented case is similar. The "only if" direction of the theorem is obvious.

Let us proof the "if" direction. Assume that $l_K[s]$ is zero co-bordant. Let $g : S_1 \to |K|$ be the co-bordism such that $g|_{\partial S_1}$ is mapped according to a given element $[s] \in B(K_0, K_2)$. We have to construct a triangulated surface, which is coloured by the vertex set of K with the given boundary colouring. We shall construct even more: a simplicial complex E together with a simplicial map $h : E \to K$ having the desired boundary behaviour.

Let us start with the compact two-dimensional manifold S_1. We may assume that S_1 does not have closed components (if it has closed components, we just remove them). Furthermore, assume that S_1 is triangulated ($S_1 = |D|$ for a simplicial complex D) in such a way that the map g is a simplicial homeomorphism on the boundary subcomplex of D.

Now we apply the simplicial approximation theorem. Let $D' = \beta^k D$ be the k-th barycentric subdivision of D and $g_2 : D' \to K$ the simplicial approximation for g. The map g_2 almost has the desired behaviour on the boundary of D_2. We have to adjust it a little to satisfy the given boundary condition. Let (xy) be a boundary edge of D. It is subdivided in 2^k edges of the complex D' by the vertex sequence $x = z_0, z_1, \ldots, z_{2^k} = y$. By the definition of simplicial approximation, the images $g_2(z_i)$ of these vertices belong to the edge $\big(g_1(x)g_1(y)\big)$ of K. Since the map g_2 is a simplicial map, we get $g_2(z_i) \in \{g_1(x), g_1(y)\}$ for all $i \in [0, 2^k]$.

Now let us add to the complex D' edges $(z_0 z_{i+1})$ and 2-simplices $(z_0 z_i z_{i+1})$ for $i \in [1, 2^k - 1]$. See Figure 14 for details (there $k = 2$). In the Figure the new edges are shown by dashed lines and the new

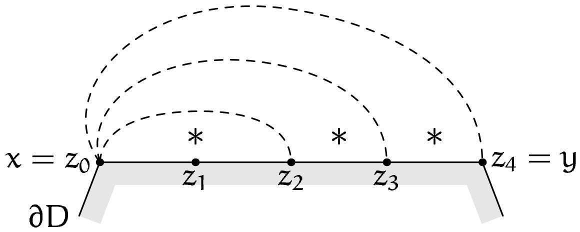


FIGURE 14. Attaching of new 2-cells to the barycentric subdivided complex.

2-cells by asterisks. Let us repeat the same construction for any boundary edge of D. Let us denote by E the simplicial complex obtained by this procedure. We now define a simplicial map $h : E \to K$ by $h(z) = g_2(z)$ for all vertices z of E. Let us observe that the image of the new edges $(z_0 z_{i+1})$ and the new simplices $(z_0 z_1 z_{i+1})$ for $i \in [1, 2^k - 1]$ are contained in the 1-simplex $\{g_1 x, g_1 y\}$ of K_0. The same observation holds for any boundary edge of D_1 hence the map $h : E \to K$ is simplicial. The theorem is proved. □

4. Homological criterion

Definition 47. Let X be a topological space, $x \in L^o X$ an oriented loop family. Let $(k, \phi) \in PX$ be an element representing it: $x = [(k, \phi)]$. For each $i \in [1, k]$ let us define the element s_i of the first homology group $H_1 X$ as the homology class of the map $\mathbb{S}^1 \to X : \zeta \mapsto \phi(i, \zeta)$. The sum of these elements $\sum_{i=1}^k s_i$ does not depend on the choice of the representing element $(k, \phi) \in x$. Let us denote this sum by $h^o_X(x)$. In the non-oriented case denote the element of $H_1(X, \mathbb{Z}/2\mathbb{Z})$ defined analogously for $y \in LX$ by $h_X(y)$.

Proposition 48. *Let X be a topological space. Let $\rho : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ be the natural projection, $\rho_* : H_1 X \to H_1(X, \mathbb{Z}/2\mathbb{Z})$ the induced map in the first homology and $\lambda_X : L^o X \to LX$ the natural projection. Then the horizontal arrows in the following diagram are surjective and the diagram commutes:*

$$\begin{array}{ccc} L^o X & \xrightarrow{h^o_X} & H_1 X \\ {\scriptstyle\lambda_X}\downarrow & & \downarrow{\scriptstyle\rho_*} \\ LX & \xrightarrow{h_X} & H_1(X, \mathbb{Z}/2\mathbb{Z}) \end{array}$$

Proposition 49. *Let* S *be a compact surface. Then the class of the boundary* $[\partial S]$ *is equal to zero in the first homology group* $H_1(S, \mathbb{Z}/2\mathbb{Z})$*. If the surface is oriented, then* $[\partial S]$ *is zero (with induced orientation) even in* $H_1(S, \mathbb{Z})$*.*

Theorem 50. *Let* X *be a topological space,* $x \in L^o X$ *a family of oriented loops. Then the element* x *is oriented zero co-bordant if and only if the equation* $h^o_X x = 0$ *holds in the homology group* $H_1 X$*.*

Proof. First note that the "only if" direction is trivial: if $f : S \to X$ is an oriented co-bordism for x, then we get $h^o_X x = f_*[\partial S] = 0$ because of $[\partial S] = 0$ in $H_1 S$.

Let us prove the "if" direction. First we may assume that the space X is arcwise connected and non-empty. Let us choose a representative $(k, \phi) \in PX$ of $x \in L^o X$. Let us choose some base point $a_0 \in X$ and connect it to the base points of the loops of our family $\phi(i, 1)$ by pathes $\gamma_i : [0, 1] \to X$: $\gamma_i(1) = \phi(i, 1)$, $\gamma_i(0) = a_0$, where $i \in [1, k]$. Consider the loop

$$\gamma = \prod_{i=1}^{k} \gamma_i \phi_i \gamma_i^{-1},$$

where the path $\phi_i : [0, 1] \to X$ is defined by $\phi_i(t) = \phi(i, e^{2\pi i \cdot t})$. The homological class $[\gamma] \in H_1 X$ is equal to $h^o_X x = 0$. Hence the homotopy class $[\gamma] \in \pi_1(X, a_0)$ is an element of the commutator $[\pi_1 X, \pi_1 X]$, according to the well-known theorem. Thus the loop γ is homotopic to a product of commutators

$$\gamma' = \prod_{j=1}^{N} \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1},$$

where α_i and β_i are some loops in X.

Let S be the direct product of the unit interval and the circle: $S = [0, 1] \times \mathbb{S}^1$ and $f : S \to X$ be a homotopy between γ and γ': see Figure 15 for details. Let us tag the boundary intervals by labels γ_i, α_j, β_j as illustrated in Figure 15. Let us identify the boundary intervals tagged by the same labels. The resulting surface S' together with the induced map $f' : S' \to X$ is the needed oriented co-bordism for the element $x \in L^o X$. The theorem is proved. □

Let us formulate a claim, which is a special case of the last theorem.

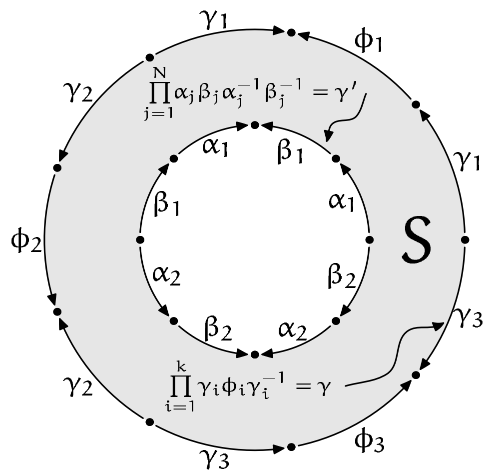


FIGURE 15. The homotopy between γ and γ' and the construction of S' (here $k = 3$, $N = 2$).

Proposition 51. *Let* c *be a null-homotopic loop in a topological space* X. *Let* $x_c \in LX$ *be the family consisting of the single loop* c. *Then* x_c *is minimal zero co-bordant of characteristic* 1.

Theorem 52. *Let* X *be a topological space,* $y \in LX$ *a family of (non-oriented) loops. The element* y *is zero co-bordant if and only if* $h_X(y) = 0$ *in the homology group* $H_1(X, \mathbb{Z}/2\mathbb{Z})$.

Proof. As in the proof of the last theorem, the "only if" direction follows from Proposition 49. Let us prove the "if" direction. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\rho} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

where the arrows are defined by $\alpha(n) = 2n$ and $\rho(m) = m + 2\mathbb{Z} = [m]_{2\mathbb{Z}}$. This short exact sequence induces the long homological exact sequence. Consider just a small part of it:

$$H_1X \xrightarrow{\alpha_*} H_1X \xrightarrow{\rho_*} H_1(X, \mathbb{Z}/2\mathbb{Z}).$$

Let us choose an element $y^o \in L^oX$ with $\lambda_X y^o = y$ (in other words, choose an arbitrary orientation on the loop family y). Because of Proposition 48 we get $\rho_* h_X^o y^o = h_X \lambda_X y^0 = h_X y = 0$. It follows $h_X^o y^o \in \ker \rho_* = \operatorname{im} \alpha_*$. Since the map $h_X^o : L^oX \to H_1X$ is surjective, we can choose an element $z \in L^oX$ with

$$\alpha_* h_X^o z = h_X^o y^o.$$

By definition of α it holds $\alpha_*(h^o_X z) = 2 \cdot h^o_X z$. Taking into account this equality we obtain

$$(*) \qquad h^o_X y^o = 2 \cdot h^o_X z$$

Define an element $x \in L^oX$ by $x = z^{-1} \sqcup z^{-1} \sqcup y^o$. It follows

$$h^o_X x = -2h^o_X z + h^o_X y^o.$$

Taking into account $(*)$ we obtain $h^o_X x = 0$. Hence by Theorem 50, the element x is oriented zero co-bordant, thus there exists an oriented surface S and a continuous map $f : S \to X$ such that $f|_{\partial S}$ is a representative for $x = z^{-1} \sqcup z^{-1} \sqcup y^o$. Now identify the components of S mapped along the loop family z^{-1}. The resulting (possibly not oriented) surface S' together with the induced map f' is the needed co-bordism for $\lambda_X y^o = y$. □

5. Metrical applications

Definition 53. Let (X, d) be a metric space (we shall allow the distance function d to have the value $+\infty$ on some pairs of points). Let I be a connected subset of $\mathbb{R}$ (the subset I can be either bounded or not, it can as well either contain its boundary or not). Let us call a continuous map $c : I \to X$ a *shortest curve* if for any $a, b \in I$ the length of the curve $c|_{[a,b]}$ is equal to the distance $d(c(a), c(b))$ and $c(a) \neq c(b)$ for $a \neq b$.

Definition 54. Let (X, d) be a metric space, I an interval in $\mathbb{R}$. Let us call a (not necessary shortest) curve $c : I \to X$ *geodesic* if for any $x \in I$ there exists a neighbourhood $U \subset I$ of x such that the curve $c|_U$ is a shortest curve.

Definition 55. A metric space (X, d) is called a *length space* if for any $x, y \in X$ with $d(x, y) < +\infty$ there exists a shortes curve connecting them.

Definition 56. Let I be a connected subset of $\mathbb{R}$. For any integer $i \in \mathbb{Z}$ denote by $I_{i,i+1}$ the (possibly empty) subset of I bounded by the consequent integers:

$$I_{i,i+1} = I \cap [i, i+1] = \{x \in I \mid i \leqslant x \leqslant i+1\}.$$

Let us call a curve $c : I \to |K|$ in a simplicial complex K *strict piecewise linear* if the following conditions hold:

(1) for every $i \in \mathbb{Z}$ there is a simplex $\sigma \in K$ such that $c|_{I_{i,i+1}}$ is an affine map from $I_{i,i+1}$ to $|\sigma|$;

(2) for every integer i from the interior of the interval I there does not exist a simplex $|\sigma| \subset |K|$ containing both $c(I_{i-1,i})$ and $c(I_{i,i+1})$.

The curve c fulfilling only the first property is called *piecewise linear*.

Definition 57. Let K be a simplicial complex, $\sigma \in K$ an n-dimensional simplex in K, $s_0, \dots, s_n$ the vertices of σ. Let $b_0, \dots, b_n \in \mathbb{R}^n$ be a generic $(n+1)$-tuple of points in $\mathbb{R}^n$ (it means, these points are not contained in an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$). Let $\iota : |\sigma| \hookrightarrow \mathbb{R}^n$ be the affine map taking s_i to b_i for $i \in [0, n]$. Let us call the distance function d_ι on $|\sigma|$ defined by $d_\iota(x, y) = d_{\mathbb{R}^n}(\iota x, \iota y)$ a *Euclidean metrics on the simplex* σ.

Definition 58. Let K be a thick N-dimensional complex. Let $D = \{d_\sigma\}_{\sigma \in K_N}$ be a family of Euclidean metrics on every top dimensional simplex of K. Let us assume that the following properties hold:

(1) the metrics of the family D are compatible: $d_{\sigma_1}(x, y) = d_{\sigma_2}(x, y)$ for every two simplicies $\sigma_1, \sigma_2 \in K_N$ and any $x, y \in |\sigma_1| \cap |\sigma_2|$;
(2) the completness property: for every infinite path $a_1, a_2, \dots$ (where a_i are vertices of K such that $\{a_i, a_{i+1}\}$ are edges of K) holds $\sum_{i=1}^{\infty} d_{\sigma_i}(a_i, a_{i+1}) = +\infty$, where σ_i is some N-simplex containing the edge $(a_i a_{i+1})$.

Such a family D is called a *piecewise Euclidean metrics* on the complex K.

Remark 59. Let K be a two-dimensional thick complex. To define a piecewise Euclidean metrics on K it is sufficient to define a positive length function ℓ on the set K_1 of the edges of K fulfilling the strict triangle inequality. In the case of a one-dimensional complex (a graph) it is sufficient to define an arbitrary positive length function (even the triangle inequality is not needed).

Now we shall introduce a metrics on the geometric realisation of a complex induced by a piecewise Euclidean metrics.

Definition 60. Let K be a thick N-dimensional complex with a piecewise Euclidean metrics $D = \{d_\sigma\}_{\sigma \in K_N}$. For any $x, y \in |K|$ denote by S_{xy} the set of all finite sequences of points in $|K|$

$$z = (x = z_0, z_1, \dots, z_{n_z} = y)$$

such that both z_{i-1} and z_i are contained in the same N-simplex $|\sigma_{i-1,i}|$ for any $i \in [1, n_z]$. Define the function $L_D : S_{xy} \to \mathbb{R}$ by

$$L_D(z) = \sum_{i=1}^{n_z} d_{\sigma_{i-1,i}}(z_{i-1}, z_i).$$

Define the distance function d_D on the set $|K|$ induced by the family D by

$$d_D(x,y) = \inf_{z \in S_{xy}} L_D(z).$$

Here are some obvious properties of the just defined metrics d_D.

Proposition 61. *Let* K *be a locally finite* N*-thick simplicial complex with a piecewise Euclidean metrics* $D = \{d_\sigma\}_{\sigma \in K_N}$. *Let* d_D *be the induced distance function on* $|K|$.

(1) *The inequality* $d_D|_{|\sigma|} \leqslant d_\sigma$ *holds for every* N*-simplex* σ.
(2) *For any point* $x \in |\sigma|^\circ$ *in the interior of an* N*-simplex* $|\sigma|$ *there exists a neighbourhood* $U \subset |\sigma|$ *of* x *with* $d_D|_U = d_\sigma|_U$.
(3) *The length of a curve in* $|K|$ *with respect to the metrics of the family* D *coincides with the length of the same curve with respect to* d_D.
(4) *The metric space* $(|K|, d_D)$ *is a length space.*
(5) *Any geodesic in the metric space* $(|K|, d_D)$ *is (up to reparametrization) a strict piecewise linear curve in the sence of Definition 56.*

Definition 62. Let K be a simplicial complex, $\sigma \in K$ a simplex. Let $L_{\sigma 0}$ be the vertex subset of K defined by

$$L_{\sigma 0} = \{x \in K_0 \mid x \notin \sigma \wedge \{x\} \cup \sigma \in K\}.$$

Let L_σ be a simplicial complex with the vertex set $L_{\sigma 0}$ defined by

$$L_\sigma = \{\sigma' \subset L_{\sigma 0} \mid \sigma \cup \sigma' \in K\}.$$

The simplicial complex L_σ is called the *link* of the simplex σ. For a vertex $x \in K_0$ let us call the link of the 0-simplex $\{x\}$ the link of the vertex x, denoted by L_x.

Proposition 63. *The following properties of a simplicial complex* K *are equivalent:*

(1) K *is locally finite;*
(2) K *is locally compact;*
(3) *the links of all non-empty simplicies in* K *are finite;*
(4) *the links of all vertices of* K *are finite.*

Now we define a piecewise Euclidean metrics for the links of vertices in a two-dimensional simplicial complex with a piecewise Euclidean metrics.

Definition 64. Let K be a thick two-dimensional complex with a piecewise Euclidean metrics $D = \{d_\sigma\}_{\sigma \in K_2}$. Let $x \in K_0$ be a vertex of K. Let us define a piecewise Euclidean metrics on the graph L_x by defining an edge length function ℓ on L_x. Let $y, z \in L_{x0} \subset K_0$ be two vertices connected by an edge $e = (yz)$ in L_x. Let α be the angle at the vertex x in the 2-simplex $\sigma = \{x, y, z\}$ of K with respect to the Euclidean metrics d_σ. Define the length of the edge e in the graph L_x by $\ell(e) = \alpha$. The piecewise Euclidean metrics on the graph defined in such a way induces a well-defined distance function on the link of the vertex x. Denote this distance function by $D|_{L_x}$.

Definition 65. Let K be a thick two-dimensional complex with piecewise Euclidean metrics. The complex K is said to be of *non-positive curvature,* if for any vertex $x \in K$ the metric graph L_x does not have a simple loop of length less than 2π.

Definition 66. Let $I \subset \mathbb{R}$ be a connected subset, $t_0 \in I$ a real number such that $t_0 < \sup I$ (it means, the interval I goes further than t_0 in the positive direction). Let $c : I \to |K|$ be a piecewise linear curve in the two-dimensional thick simplicial complex K with piecewise Euclidean metrics $D = \{d_\sigma\}_{\sigma \in K_2}$. Let $x = c(t_0)$ be a vertex of K. Let $\sigma = \{x, y, z\}$ be a 2-simplex containing the image of $c|_{[t_0, t_0+\epsilon)}$ for some small $\epsilon > 0$. For $\nu \in \{y, z\}$ let α_ν denote the angle between the edge $(x\nu)$ and the curve $c|_{[t_0, t_0+\epsilon)}$ with respect to d_σ. Obviously we have $\alpha_y + \alpha_z = \ell(yz)$, where ℓ is the length function on the link L_x induced by D as defined above. Let $\dot{c}(t_0)$ be the point on the edge (yz) of the link L_x at the distance α_y from $y \in L_x$ and α_z from $z \in L_x$ — see Figure 16 for details. We shall call the element $\dot{c}(t_0) \in L_x$ the *direction* of the curve c at t_0. In non-ambiguous context we shall use the notation c_x (recall that $x = c(t_0)$) instead of $\dot{c}(t_0)$.

Definition 67. Let the interval I and the complex K be as above. Let $t_0 \in I$ be a point with $t_0 > \inf I$ such that $x = c(t_0)$ is a vertex of K. Define the *inverse direction* $-\dot{c}(t_0)$ by $-\dot{c}(t_0) = \dot{\tilde{c}}(0)$, where the curve $\tilde{c}$ is defined by $\tilde{c}(t) = c(t_0 - t)$. We shall use the notation $-c_x$ as well where non-ambiguous.

Remark 68. If t_0 is a point in the interior of the interval I, then both directions c_x and $-c_x$ are defined.

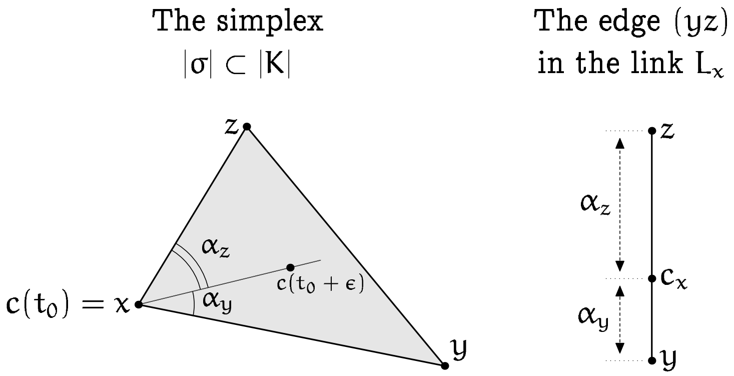


FIGURE 16. Definition of the direction $c_x \in L_x$ of the curve c.

Proposition 69. *Let* K *be a two-dimensional thick complex with piecewise Euclidean metrics,* $c : I \to |K|$ *a strict piecewise linear curve,* $t_0 \in I^\circ$ *a point in the interior of the interval* I *such that* $x = c(t_0)$ *is a vertex of* K*. Then* c *is a geodesic in a small neighbourhood of* t_0 *if and only if* $d_{L_x}(c_x, -c_x) \geqslant \pi$*.*

Definition 70. Let K be a two-dimensional thick complex with piecewise Euclidean metrics, $c : I \to |K|$ a piecewise linear curve, $t_0 \in I^\circ$ a point in the interior of the interval I such that $x = c(t)$ is a vertex of K. Let us call the real number $k(t_0) = k_x := \pi - d_{L_x}(c_x, -c_x)$ the *curvature* of c at the point x. The curvature k_x is non-positive in case of a geodesic c.

Let K be a thick simplicial complex of dimension at least 2. Let (S, Γ, f) be a minimal triangulated (K_0, K_2)-coloured surface. As we have seen in Section 2, the surface S can be described as the geometric realisation of some simplicial complex D

$$(S, \Gamma, f) = (|D|, |D|_1, f)$$

and the colouring map $f : D_0 \to K_0$ is a simplicial map (because the colouring is (K_0, K_2)-admissible), which takes any three vertices of a 2-simplex of D to three different vertices of the complex K.

Now let K be a complex with a piecewise Euclidean metrics and let $\ell_K : K_1 \to \mathbb{R}$ be the edge length function. The colouring map f induces a function $\ell_D : D_1 \to \mathbb{R}$ by setting $\ell_D(xy)$ to be equal to the length $\ell_K(fx, fy)$ of the image in K_1 under the map f. Because of the property (3) in Theorem 27 the number $\ell_D(xy)$ is positive and the strict triangle inequality holds. In such a way we introduce a piecewise Euclidean metrics on the minimal surface $S = |D|$.

Now let K be a 2-dimensional complex of non-positive curvature. Because of the property (5) in Theorem 27 the induced piecewise Euclidean metrics on the minimal surface $S = |D|$ is also of non-positive curvature. Let $x \in \partial S$ be a vertex of the complex D. There exists a neighbourhood of x, which is homeomorphic to $\mathbb{R}^2_+$. Thus the link L_x is homeomorphic to a compact interval in $\mathbb{R}$. Let ℓ_x be the length of this link with respect to the link metrics as described for two-dimensional complexes in Definition 64. Let the vertices $x_1, x_2 \in \partial S$ be the neighbours of x on the boundary of the complex D. Let c be the curve connecting x_1 to x_2 along the edges $(x_1 x)$ and $(x x_2)$. In this setting we have the obvious inequality

$$(**) \qquad \ell_x \geqslant d_{L_{fx}}((f \circ c)_{fx}, -(f \circ c)_{fx}),$$

where the right hand side is the distance between the directions of the curve $f \circ c$ in the complex $|K|$.

Theorem 71. *Let* D *be a finite two-dimensional simplicial complex with a piecewise Euclidean metrics of non-positive curvature. Let the geometric realisation* $|D|$ *of this complex be a connected manifold with non-empty boundary. Let* $L > 0$ *be the number of boundary edges in* $|D|$*. Let* R *be the sum of the link lengths* ℓ_x *taken over all boundary vertices* x *of* $|D|$*. Then the following inequality holds:*

$$\frac{R}{\pi} \leqslant L - 2 \cdot \chi|D|.$$

Proof. Let w be the sum of the lengths of all edges of the links of all vertices in D (of the inner as well as of the boundary vertices). Let $a_2 = \#D_2$ be the number of 2-simplicies of D. Because the angle sum in any Euclidean triangle is equal to 180°, we have

$$(1) \qquad w = \pi \cdot a_2.$$

Let $a_0 = \#D_0 - L$ be the number of inner vertices of D. Since the metrics on $|D|$ is of non-positive curvature, the length of any non-trivial circle in any inner link is at least 2π, so we have

$$(2) \qquad w \geqslant 2\pi \cdot a_0 + R.$$

Let $a_1 = \#D_1 - L$ be the number of inner edges in D. Since

$$a_0 - a_1 = (\#D_0 - L) - (\#D_1 - L) = \#D_0 - \#D_1$$

we have

$$(3) \qquad \chi|D| = \#D_0 - \#D_1 + \#D_2 = a_0 - a_1 + a_2.$$

Obviously we have

$$3 \cdot a_2 = 2 \cdot a_1 + L, \tag{4}$$

because any triangle has exactly three sides, any inner edge is contained in exactly two triangles, and any boundary edge is contained in only one triangle. Combining (1) and (2), we get

$$2\pi \cdot a_0 + R \leqslant \pi \cdot a_2.$$

Multiplying both sides by $1/\pi$, we get

$$2a_0 + R/\pi \leqslant a_2.$$

Taking into account (3) and substituting $\chi|D| - a_2 + a_1$ for a_0, we obtain

$$2 \cdot \chi|D| - 2a_2 + 2a_1 + R/\pi \leqslant a_2.$$

Adding $2a_2$ to both sides, we get

$$2 \cdot \chi|D| + 2a_1 + R/\pi \leqslant 3a_2.$$

Combining this with (4), we get

$$2 \cdot \chi|D| + 2a_1 + R/\pi \leqslant 2a_1 + L.$$

Substructing $2a_1$, we obtain

$$2 \cdot \chi|D| + R/\pi \leqslant L,$$

which proves the theorem. □

Corollary 72. *Let* K *be a two-dimensional simplicial complex of non-positive curvature with a piecewise Euclidean metrics. Let* $[s] \in B(K_0, K_2)$ *be a boundary condition (see Definition 19). Let* $l_K[s]$ *be the corresponding loop family as in Definition 40. Assume that* $l_K[s]$ *is minimal zero co-bordant of characteristic* χ *as in Definition 45. Let* κ *be the sum of the curvatures of the loops of* $l_K[s]$ *taken over all vertices of this loop family. Then we have*

$$\kappa \geqslant 2\pi \cdot \chi.$$

Proof. Let D be a minimal surface of Eulerian characteristic χ with the given boundary condition equipped with the piecewise Euclidean metrics of non-positive curvature induced by the colouring map. Let R and L be as in Theorem 71. Then we have

$$\kappa = \sum_{x \in \partial|D|} (\pi - w_x),$$

where $w_x = d_{L_x}(c_x, -c_x)$ is the direction distance of the boundary loop c at point x. Because of the inequality $(**)$ on page 27, the number R fulfills the inequality

$$R \geqslant \sum w_x. \tag{5}$$

By the definition of L we obtain

$$\kappa = \pi L - \sum w_x.$$

Taking into account the inequality (5) we get

$$\kappa \geqslant \pi L - R.$$

On the other hand Theorem 71 implies

$$\pi L - R \geqslant 2\pi \cdot \chi.$$

Combining the last two inequalities we obtain $\kappa \geqslant 2\pi \cdot \chi$. □

Corollary 73. *In the setting of Theorem 71 assume that the boundary of* $|D|$ *is geodesic in all up to* g *boundary vertices. Then*

$$g \geqslant 2\chi|D|$$

and this inequality is strict for $g > 0$.

Proof. The boundary is geodesic in $L-g$ vertices. Combining Proposition 69 with the inequality $(**)$ on page 27 we obtain that the link lengths in these vertices are at least π, hence

$$R \geqslant (L-g)\pi + R',$$

where R' is the sum of the link lengths taken over the remaining g boundary vertices. The link lengths in the remaining g vertices are positive, hence $R > (L-g)\pi$ if $g > 0$. Dividing the both sides of the last inequality by π, we obtain

$$\frac{R}{\pi} \geqslant L - g.$$

On the other hand Theorem 71 implies

$$\frac{R}{\pi} \leqslant L - 2\chi|D|,$$

hence

$$L - 2\chi|D| \geqslant R/\pi \geqslant L - g.$$

The claim of the corollary follows. □

Corollary 74. *Let* K *be a two-dimensional simplicial complex of non-positive curvature with a piecewise Euclidean metrics. Let* c *be a null-homotopic simple edge path consisting of* g *geodesic seqments. Then* $g \geqslant 3$*.*

Proof. Just apply Corollary 73 for a minimal surface spanning the curve c. Because the curve c is null-homotopic, the minimal surface will be homeomorphic to a disk, and therefore its Eulerian characteristic is equal to 1. Corollary 73 implies now $g \geqslant 2$. Moreover, g is positive, hence Corollary 73 implies $g > 2$, hence $g \geqslant 3$. □

Now we can give a proof of the following well-known fact.

Corollary 75. *In a two-dimensional simplicial complex with a piecewise Euclidean metrics of non-positive curvature any two geodesics connecting given points* x *and* y *either coincide or are non-homotopic.*

Proof. The claim follows directly from the last corollary. □

Epilogue

This work is arisen from a failed attempt to prove the following conjecture inspired by a theorem claimed by M. Gromov in [Gro87] and further discussed by T. Delzant.

Conjecture 76. *Let* K *be a locally compact thick two-dimensional simplicial complex with a piecewise Euclidean metrics of non-positive curvature. Let* c *be a closed geodesic in* $|K|$ *(it means,* $c : \mathbb{R} \to |K|$ *is a geodesics, and there exists a positive number* $M \in \mathbb{R}$ *fulfilling* $c(t) = c(t+M)$ *for all* $t \in \mathbb{R}$*), and let* $x \in K_0$ *be a vertex on the geodesic* c *such that the curvature* k_x *at the point* x *of* c *(in the sence of Definition 70) is strictly negative:* $k_x \leqslant -\epsilon < 0$ *(or, equivalently:* $d_{L_x}(c_x, -c_x) \geqslant \pi + \epsilon > \pi$*).*

Then there exists a natural number $m \in \mathbb{N}$ *such that the minimal normal subgroup of the fundamental group* $\pi_1|K|$ *containing the* m*-th power of the loop* $c_0 = c|_{[0,M]}$ *is a free group generated by some conjugates* $a_i[c_0]^{mk_i}a_i^{-1}$ *of powers of* $[c_0]^m$ *(where* $a_i \in \pi_1|K|$*,* $k_i \in \mathbb{N}$*).*

The proof idea is based on the construction of a minimal surface in the complex $|K|$. Consider a "non-trivial" relation

$$(*) \qquad \prod_{i=1}^{N} a_i[c_0]^{mk_i}a_i^{-1} = 1 \in \pi_1|K|.$$

Let γ be a lift of the curve $\prod a_i c_0^{mk_i} a_i$ into the fundamental covering $q : X \to |K|$. The space X is again a simplicial complex with piecewise Euclidean metrics of non-positive curvature induced by the covering map. Since the curve $q \circ \gamma$ is null-homotopic in $|K|$, the start and the end points of γ coincide. Now assume that the relation $(*)$ can be choosen in such a way that the curve γ does not have self-intersections and the elements a_i are represented by geodesics. Because of Proposition 51, there exists a minimal surface of Eulerian characteristic 1 in X spanning a polygon consisting of $3N$ geodesic segments: for any $i \in [1, N]$ we have the liftings of three curves: of $c_0^{mk_i}$ and of geodesic representatives of a_i and a_i^{-1}. Apply Corollary 72: $\kappa \geqslant 2\pi$, where κ is the sum of curvatures of the boundary curve. Since the curvature is non-positive in all inner vertices of geodesic segments, and it is at most $-\epsilon$ in at least $m \sum k_i \geqslant mN$ points, it holds:

$$2\pi \leqslant \kappa \leqslant -\epsilon \cdot mN + 3N\pi$$

(where the last term $3N\pi$ is the upper boundary for the curvature sum taken over all $3N$ vertices, in which the different geodesic segments are connected). It follows

$$\epsilon \leqslant \frac{1}{m} \cdot \frac{3N-2}{N} \cdot \pi = O(1/m)$$

This inequality can not hold for large m, which would prove that there are no "non-trivial" relations as in $(*)$.

Alas the author does not know how to define the non-triviality of a relation in such a way that the injectivity assumption holds.

The metric applications of Section 5 can not be generalized for the case of complexes of higher dimension. The reason is: the length of a loop in a vertex link of a higher-dimensional complex of non-positive curvature is not necessarily bounded from below by 2π. Therefore the minimal surface in such a complex does not inherit the non-positive curvature anymore (which is a necessary condition used in the proof of Theorem 71 — see the inequality (2) on page 27).

To visualize this situation consider the following example. Let σ_2 be a 2-simplex in $\mathbb{R}^3$. Let $K_k = \beta^k \sigma_2$ be the k-th barycentric subdivision of the simplex σ_2 and let K_{k0} be the vertex set of K_k. Let $a \in \mathbb{R}^3$ be an arbitrary point not contained in the plane of the simplex σ_2.

Let the complex $\tilde{K}_k$ be defined as

$$\tilde{K}_k = \left\{ \sigma \subset K_{k0} \cup \{a\} \mid \sigma \backslash \{a\} \subset K_k \right\}$$

— see Figure 17. The bottom face of the pyramid $\tilde{K}_k$ contains $O(6^k)$

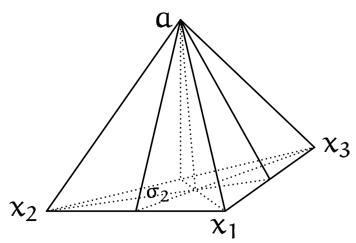


FIGURE 17. The pyramid $\tilde{K}_k$ constructed as a cone over the k times barycentric subdivided simplex $\sigma_2 = \{x_1, x_2, x_3\}$, here $k = 1$.

edges. The other three faces — only $O(2^k)$ edges. For large values of k it follows that the bottom face is not a minimal surface spanning ist own boundary anymore. The minimal surface spanning the boundary of the bottom face consists of the other three faces of the pyramid. This surface is not of non-positive curvature anymore because the link loop of the vertex a is shorter than 2π.

References

[Gro87] Mikhael Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263.

Contents

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